A HIERARCHICAL ALGORITHM FOR FAST BACKPROJECTION IN HELICAL CONE-BEAM TOMOGRAPHY

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ABSTRACT

Existing algorithms for exact helical cone beam (HCB) tomographic reconstruction involve a 3-D backprojection step, which dominates the the computational cost of the algorithm. We present a fast hierarchical 3-D backprojection algorithm, generalizing fast 2-D parallel beam and fan beam algorithms, which reduces the complexity of this step from $O(N^4)$ to $O(N^3 \log N)$, greatly accelerating the reconstruction process.

1. INTRODUCTION

Helical cone-beam tomography has several advantages over traditional two dimensional tomographic imaging, including decreased scanning times and increased x-ray source utilization. However, image reconstruction from cone beam projections relies on inversion formulas [3], [2] of higher complexity than those found in two dimensional tomography. These algorithms consist of individually "filtering" the cone beam projections followed by a backprojection over the image volume. This 3-D backprojection has complexity of $O(N^3P)$ for reconstruction of an $N \times N \times N$ voxel image from P projections. Generally $P = O(N)$, which results in an $O(N^4)$ operation and accounts for a large amount of the computation in the reconstruction process.

Several fast algorithms for backprojection in two dimensional tomography exist. Algorithms based on hierarchical decomposition reduce the complexity of the backprojection operation by successively subdividing the reconstruction area into smaller nonoverlapping regions. As the region size decreases, the number of projections necessary for accurate reconstruction also decreases. The number of projections can then be reduced, which reduces the computational complexity. This hierarchical decomposition of the backprojection operation initially developed for 2-D parallel beam [1], was extended to fan beam [4] and 3-D cone beam with a circular trajectory (CCB) [5]. Here we extend the method to the more general divergent-beam geometry of a helical trajectory. HCB does not suffer from the inherent artifacts present in CCB but carries a higher computational cost. The fast CCB method decomposes the volume only in the x and y dimensions. Here we fully decompose the volume along all three dimensions. This is important for accurate reconstruction with a helical trajectory, and for extensions to an arbitrary trajectory.

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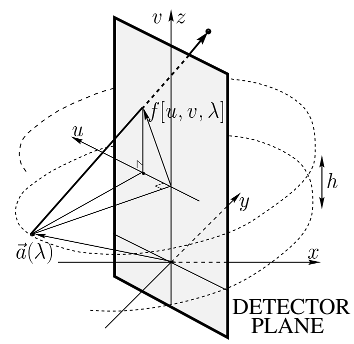

Fig. 1. Helical Cone-Beam Acquisition

2. ALGORITHM DESCRIPTION

2.1. 3-D Cone Beam Backprojection

Figure 1 shows the setup for helical cone beam projection data acquisition. An x-ray source is placed at equally spaced intervals along a helical trajectory parameterized by $\vec{a}(\lambda) = (R\,cos\lambda, R\,sin\lambda, \frac{h\lambda}{2\pi})$ where R is the distance between the source and the z-axis and h is the pitch of the helix. The detector plane is assumed to contain the z-axis and be perpendicular to $\vec{a}(\lambda)$. After the filtering step is completed for each cone beam projection, the filtered projection data $\tilde{f}[u, v, \lambda]$

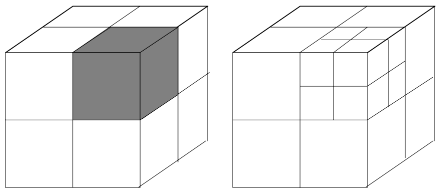

Fig. 2. (a)Octant Decomposition, (b)Recursive Decomposition

is backprojected onto the image volume according to

$$I(\vec{x}) = \sum_{\lambda} \frac{R\sqrt{R^2+h^2}}{|<\vec{x}-\vec{a}(\lambda),(cos\lambda, sin\lambda, 0)>|^2} \times \tilde{f}(P_\lambda^U(\vec{x}), P_\lambda^V(\vec{x}), \lambda) \quad (1)$$

where $< .,. >$ represents an inner product and the detector plane coordinate $[P_\lambda^U(\vec{x}), P_\lambda^V(\vec{x})]$ is the projection of the point $\vec{x}$ from source position $\vec{a}(\lambda)$. Each object point requires a sum over all the projections, leading to an $O(N^3P)$ complexity for the entire image. This reconstruction of the object directly from the filtered projection set using (1) will be referred to as the conventional backprojection algorithm for the remainder of this paper.

2.2. Object Decomposition

Hierarchical algorithms are based on the principle that the number of projections needed to accurately reconstruct an image is proportional to the size of the image. The filtered projection data to be backprojected is denoted by $\tilde{f}(\vec{t}, \theta)$, where $\vec{t}$ is the (possibly two dimensional) detector position and θ indexes the source position. Studies on the spectral support of $\tilde{f}$ state that the number of projections required for an object with support restricted to a radius N from the origin is proportional to N. For an origin centered object, the number of required projections is $P = kN$ for some constant k. If $P = \ell kN$ for some integer $\ell > 1$, then $\tilde{f}(\vec{t}, \theta)$ can be decimated by a factor of ℓ in θ without any loss in reconstructed image quality. This is implemented via an angular filter followed by downsampling

$$\tilde{f}_D[\vec{t}, \theta] = (\tilde{f} * G)[\vec{t}, \ell\theta] \quad (2)$$

where G is a 1-D low-pass filter and the convolution occurs only along the θ dimension. The computation involved in the backprojection of the set $\tilde{f}_D$ is reduced by $1/\ell$ over backprojection of the set $\tilde{f}$, but still accurately reconstructs the desired region.

The reconstruction volume is comprised of 8 octants, each of which has dimensions reduced by half relative to the original problem size (Figure 2a). Therefore, each octant should require only half the number of projections needed for reconstruction of the larger volume. However, the decimation of projections described previously is only applicable for objects centered at the origin.

The method for achieving a reduction in the number of projections is motivated by the procedure developed for fast 2-D fan beam [4] and 3-D circular trajectory cone beam backprojection [5]. For helical cone beam data, this process is described as

$$\tilde{f}_D = \Delta_c^- \Psi_{\downarrow 2} \Delta_c^+ \tilde{f} \quad (3)$$

The operator Δ_c shifts the projections relative to the projection of the center $\vec{c}$ of the decomposition region

$$\Delta_c^{\pm} f(u, v, \lambda) = f(u \pm P_\lambda^U(\vec{c}), v \pm P_\lambda^V(\vec{c}), \lambda) \quad (4)$$

and the operator $\Psi_{\downarrow 2}$ decimates the shifted projections by first convolving with a 1-D low-pass filter along the λ dimension and then downsampling.

In this procedure, the projection data is first shifted so that the projection of the center of the region lies at the origin of the detector plane. In the parallel beam case, this shift exactly represents moving the subregion to the origin. In the case of divergent beam tomography, this shift is approximate but successfully reduces the angular bandwidth of the projection data and allows for decimation. After decimation, the remaining projections are shifted back to their original positions and then the set $\tilde{f}_D$ is backprojected over the particular subvolume using (1).

2.3. Fast Hierarchical Algorithm

In decomposing the object into 8 octants and decimating the projections using the above procedure, the amount of work spent doing backprojection has been reduced by a factor of 2, but the $O(N^4)$ complexity remains. The reduction in complexity is achieved by applying this decomposition recursively, as shown in Figure 2b, yielding successively smaller subvolumes. After every subdivision, a reduction of the number of projections is possible with additional shifting and decimation using (3). This division continues until the subvolume has reached a minimum size, such as $N = 1$, which involves decomposing $log_2 N$ times. After $log_2 N$ stages there are $O(\frac{P}{2^{log_2 N}}) = O(\frac{P}{N}) = O(1)$ projections available for reconstructing each voxel, so the entire object can now be reconstructed in $O(N^3)$.

The amount of work involved in the shifting and decimation step is proportional to the number of projections and the size of each projection. A subvolume which has been halved in each dimension will have the size of its projections reduced by a factor of 4. At stage k in the decomposition, there will be 8^k subvolumes with $\frac{P}{2^k}$ projections each of size proportional to $\frac{N^2}{4^k}$. Therefore the total work

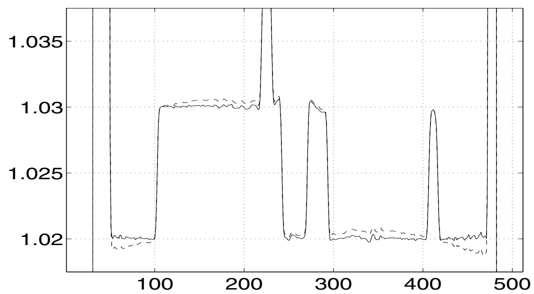

Fig. 3. Hierarchical reconstructions with (solid) and without (dashed) weighting function

involved in decimating the projections for all 8^k subvolumes is $O(N^2P)$, resulting in roughly constant work per stage. For log_2N stages, the overall complexity becomes $O(PN^2 log\, N)$. Replacing $P = O(N)$ yields the final complexity of the algorithm to be $O(N^3 log\, N)$, which is a reduction in order from the $O(N^4)$ complexity of (1).

2.4. Weighting Function

An enhanced implementation of the decimation procedure includes a weighting of the projection data before and after angular decimation.

$$\tilde{f}_{ds} = \Delta_c^- W_2[c, \lambda] \Psi_{\downarrow 2} W_1[c, \lambda] \Delta_c^+ \tilde{f} \tag{5}$$

This weighting is a function of the source position and the center of the decomposition region $\vec{c}$. The effect of this weighting is to further compress the spectrum of the projection data prior to decimation. In this case we select W_1 to be the weighting used in the backprojection equation (1) and W_2 to be its inverse

$$W_1[c, \lambda] = \frac{1}{W_2[c, \lambda]} = \frac{1}{| < \vec{c} - \vec{a}(\lambda), (cos\lambda, sin\lambda, 0) > |^2} \tag{6}$$

It can be shown that this weighting minimizes the bandwidth of the projections for points in the neighborhood of $\vec{c}$. Figure 3 shows slices through a reconstruction that includes the weighting operation and a reconstruction that does not use the weighting operation. This weighting helps maintain the accuracy of the hierarchical reconstruction. This weighting can be absorbed into the shifting operations, resulting in a minimal impact on the computation required by the algorithm.

2.5. Tunable Parameters

The above algorithm involves shifting and angular filtering, which must be done accurately for good object reconstruction. Longer filters give better reconstructions but also require more computation, resulting in a trade off between speed and accuracy.

When the projection data is oversampled, shorter filters will provide sufficient accuracy. A simple way to increase the oversampling of the projection data with respect to λ is to perform a single level decomposition without performing the decimation step, or equivalently decomposing the object into 64 subregions for the first stage. Each subvolume then has a projection set sampled at twice the previous rate, allowing for shorter angular filters. The number of times a non-decimated decomposition is performed is referred to as the holdoff factor. The drawback is that since each octant now has twice as many projections, the algorithm must perform twice as much work than if no holdoff were used.

3. RESULTS

Simulated projection data for the standard 3-D Shepp-Logan phantom was generated using analytical expressions for the cone beam projections. The data was then filtered [2] followed by backprojection onto a 512^3 voxel grid using both the conventional and hierarchical backprojection methods, with a holdoff factor of 1 in the hierarchical method. Figures 4-6 compare slices through the reconstructed images, and show good agreement. A tight range of values is displayed in order to show the differences between the two algorithms. Backprojection using the conventional algorithm required 67.6 hours whereas the hierarchical algorithm required 4.7 hours, a speedup of 14.4. Further optimization in the code implementations will change the relative speedup between the two. However, the hierarchical algorithm has several parameter selections (filter type, filter length, holdoff factor) that can be adjusted along with code optimizations, which may result in an even faster reconstruction using the hierarchical method.

Another method of comparison is the total number of floating point multiplies and divides in the respective algorithms. As a metric, we define the Multiplicative Effort as the total number of floating point multiplies and divides performed in an algorithm. This aids in isolating the data-driven component of the algorithm from other issues such as processor speed and memory size that are relevant to the execution time. For the $N = 512$ case, the conventional algorithm requires a Multiplicative Effort of $2.27 \cdot 10^{12}$, whereas the hierarchical algorithm requires a Multiplicative Effort of $1.54 \cdot 10^{11}$, a ratio of 14.8. The measured speedup in execution time was similar to this ratio.

The Multiplicative Effort for the conventional algorithm and the hierarchical algorithm are compared in figure 7 for a range of problem sizes of $N = 32$ to $N = 1024$. This plot shows the reduction in order of the hierarchical algorithm which indicates the computational savings become even more substantial as the object size increases.

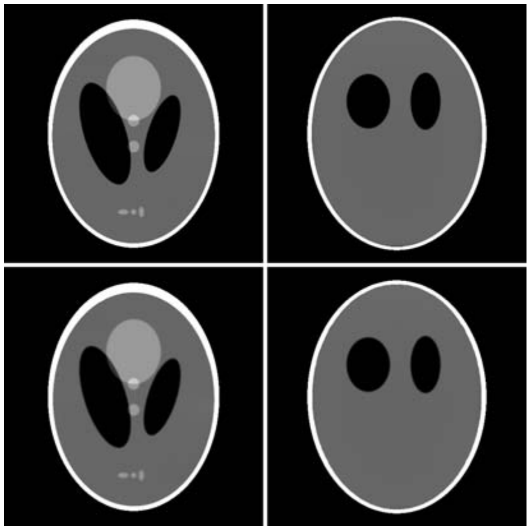

Fig. 4. Reconstruction slices y = 66 (left column) and z = 0 (right column) for conventional (top row) and hierarchical (bottom row) backprojection methods. The grayscale window is [1 1.05].

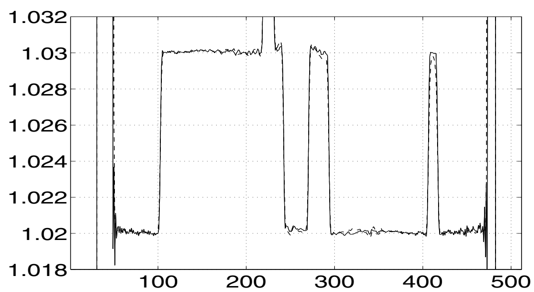

Fig. 5. Cut (x=0,y=66) through conventional (solid) and hierarchical (dashed) reconstructions

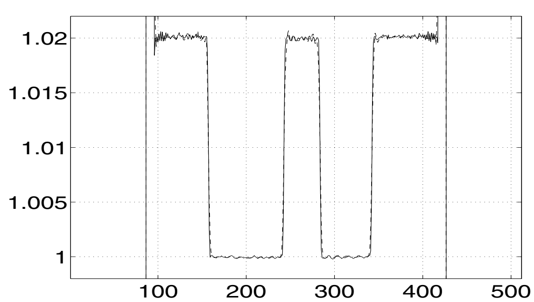

Fig. 6. Cut (y=-63,z=0) through conventional (solid) and hierarchical (dashed) reconstructions

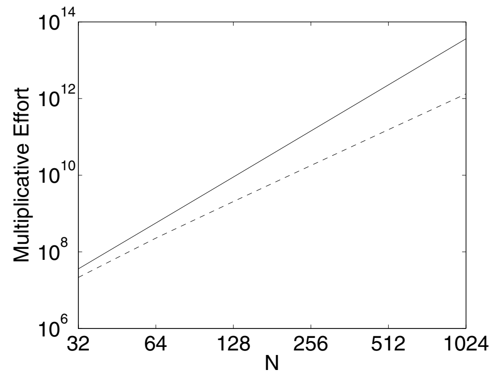

Fig. 7. Multiplicative Effort Plot: Solid = Conventional, Dashed = Hierarchical

4. CONCLUSION

In this paper, we described a fast hierarchical algorithm for 3-D backprojection in helical cone-beam tomography. The algorithm gives comparable reconstructions to the slow, exact backprojection, and results in a speedup of over an order of magnitude due to its reduction in the computational complexity.

5. REFERENCES

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