Sampling Requirements for Circular Cone Beam Tomography

Jeffrey Brokish[1] and Yoram Bresler[2]
[1]InstaRecon, Inc., Urbana, Illinois, USA
[2]Coordinated Science Lab and Dept. ECE,
Univerisity of Illinois at Urbana-Champaign, Urbana, Illinois, USA

Abstract— **The sampling requirements of tomographic projection data are determined by the set of frequencies occupied by the Fourier Transform of the projection data. This region of essential support has been analyzed for various 2-D geometries. The general case of 3-D cone beam projections, using a 2-D detector array, is mainly unexplored. In this paper, we consider the 3-D circular cone-beam (CCB) scan. An analysis of the essential support of CCB projection data provides the sampling requirements for detector spacing, row spacing, and projection count (views).**

I. Introduction

The sampling requirements of tomographic projection data are determined by the set of frequencies occupied by the Fourier Transform of the projection data. Since the object being scanned is spatially limited, it's spectral support (and the spectral support of its projections) cannot be strictly bandlimited. Therefore it is common to refer to the *essential support* of the spectral content of the object and its tomographic projections. The essential support is defined as the region outside which the magnitude of the spectral data decays exponentially.

The cases of 2-D parallel and fan beam data have been previously studied [1] and the essential support has been shown to have the shape of a bow tie, which is a function of the size of the object and the geometry of the scan. The special case of 3-D scanning with a 1-D detector array has also been analyzed [2]. Here we consider the 3-D circular cone-beam (CCB) scan, which has not been previously explored. An analysis of the essential support of CCB projection data provides the sampling requirements for detector spacing, row spacing, and projection count. This ensures that the continuous projection data can be recovered from its sampled version. Because the CCB geometry does not provide a complete data set, this can not guarantee artifact free 3-D reconstruction. However, it does ensure that the reconstructed volume will be free of artifacts due to undersampled data.

II. Definitions

A CCB scan is shown in Figure 1, with cone beam projections $g(u, v, \beta)$ collected on a flat panel detector index by horizontal detector offset u, and vertical detector offset v.

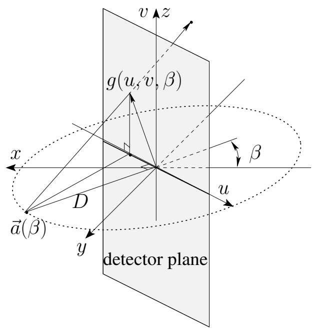

Fig. 1. Circular Cone Beam Scanning

The x-ray source rotates in the x-y plane at a distance D from the origin, with source angle denoted by β. The detector plane rotates with the source and is mathmatically defined as containing the z axis. Define the cone projection of point $\vec{x}$ from source position β as the detector plane position $(u_x(\beta), v_x(\beta))$.

The object being scanned is assumed to be bounded by a cylinder of radius R and height Z, which is a typical object support for CCB scanning. It is also essentially supported on a sphere of radius Ω in the Fourier domain.

To assess the sampling requirements of the projection data $g(u, v, \beta)$ for this object, the 3-D Fourier Transform is taken. CCB data is periodic in β, so the bandwidth analysis computes the Fourier series coefficients for integer k,

$$G(\omega_u, \omega_v, k) = \iiint g(u, v, \beta) e^{-j(\omega_u u + \omega_v v + k\beta)} \, d\beta \, dv \, du \tag{1}$$

Since the object is spatially limited, the cone beam projections will have finite support, implying that $G(\omega_u, \omega_v, k)$ will not be strictly bandlimited. Therefore, the bandwidth analysis involves finding the essential support of (1), which is defined as the region outside which the function decays exponentially.

III. Determining the Essential Support of G

A bound for the essential support of G is determined by an analysis of ω_u and ω_v jointly, and k independently. These two boundaries enclose a volume that serves as an estimate of the essential support of $G(\omega_u, \omega_v, k)$.

Supported in part by NSF grants Nos. CCR 02-09203 and OII-0610904, and NIH grants Nos. 1 R43 EB005067-01 and 1 R43 EB005576-01

A. Bounds in ω_u, ω_v

Bounds on the support of the projection data in ω_u and ω_v can be determined through an examination of the imaging geometry. For simplicty, assume the source is located on the x-axis. Consider a plane $\Pi_x(y,z)$ parallel to the detector plane. The Fourier Transform (FT) of this 2-D slice of the object is related to the 3-D FT of the object

$$\mathcal{F}^2(\Pi_x)(\omega_y,\omega_z) = \int F(\omega_x,\omega_y,\omega_z)e^{-j\omega_x x}dx \tag{2}$$

This 2-D FT will vanish for $\omega_y^2 + \omega_z^2 > \Omega^2$ because, by assumption, $F(\omega_x,\omega_y,\omega_z) \sim 0$ for $\omega_x^2+\omega_y^2+\omega_z^2 > \Omega^2$. This implies that the FT of the 2-D slice is essentially supported on a disk of radius Ω in ω_y, ω_z.

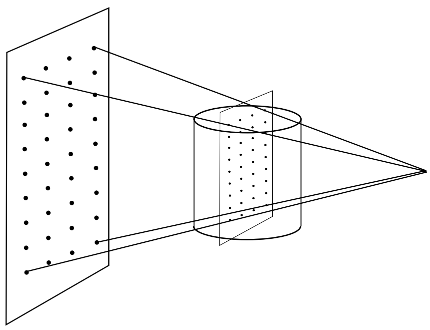

Fig. 2. Magnification property of cone beam projections

The cone beam projection acts as a magnifier of this slice (Figure 2), resulting in the data sampled on the detector plane

$$g(u,v) = \Pi_x\left(\frac{u(D-x)}{D}, \frac{v(D-x)}{D}\right) \tag{3}$$

The scaling property of the Fourier Transform implies that $G(\omega_u,\omega_v)$ will be essentially supported on a disk of radius $\Omega D/(D-x)$. The total projection is the integral over all planes intersecting the cylinder. The largest region support of $G(\omega_u,\omega_v)$ will result from $x=-R$, providing the bound

$$\omega_u^2+\omega_v^2 \le \frac{\Omega(D+R)}{D} \tag{4}$$

Since the object's spatial and spectral support are rotationally invariant with source position, this holds for all source positions.

B. Bounds in k

This analysis begins by considering the projection of a single point inside the object. The union of the spectral supports of the projections of all points in the object form a bound for the essential support of (G). Omitting a weighting function in β unimportant to the bandwidth analysis [3], the cone beam projections of a point $\vec{x}$ are

$$g(u,v,\beta) = \delta(u-u_x(\beta), v-v_x(\beta), \beta) \tag{5}$$

Expressing the point $\vec{x}$ in cylindrical coordinates (R,ψ,z), the detector position of the projected point can be written

$$u_x(\beta) = \frac{DRsin(\beta+\psi)}{D-Rcos(\beta+\psi)} \quad v_x(\beta) = \frac{Dz}{D-Rcos(\beta+\psi)} \tag{6}$$

Substituting this into (1) yields

$$G_{\vec{x}}(\omega_u,\omega_v,k) = \int_0^{2\pi} e^{j(\epsilon(\beta)-k\beta)}d\beta \tag{7}$$

where

$$\epsilon(\beta) = \omega_u u_x(\beta) + \omega_v v_x(\beta) \tag{8}$$

Since the integral in (7) covers the range $(0,2\pi)$, a simple change of variables in β will remove ψ without modifying the essential support of G.

Previous work [3] examining the essential support of integrals of the form (7) expressed the bounds of the essential support in terms of the Fourier series coefficients of $\epsilon(\beta)$. The structure of the expressions in (6) allows for these coefficients to be computed explicitly. Consider the function

$$h(\beta) = (1-\alpha cos(\beta))^{-1} \tag{9}$$

for $\alpha = R/D \in [0,1)$. The complex Fourier series coefficients c_n of $h(\beta)$ can be determined by a contour integral and the residue theorem yielding a closed form expression

$$c_n = S\gamma^n \quad \gamma = \frac{a}{1+\sqrt{1-\alpha^2}} \quad S = \frac{2}{\sqrt{1-\alpha^2}} \tag{10}$$

A similar derivation with $sin(\beta)$ in the numerator of (9) yields coefficients

$$c_n = \frac{j}{2}(\gamma-\gamma^{-1})S\gamma^n \tag{11}$$

with S and γ defined as before. Therefore the complex Fourier series coefficients of $\epsilon(\beta)$ can be expressed as

$$c_n = \left(\frac{j}{2}\omega_u R(\gamma-\gamma^{-1}) + \omega_v z\right)S\gamma^n = M\gamma^n \tag{12}$$

or equivalently, $\epsilon(\beta)$ can be written

$$\epsilon(\beta) = |M|\sum_n \gamma^n cos(n\beta + \angle M) \tag{13}$$

It is easy to see that functions of the form in (13) will result in an essential support width that is linear with $|M|$. It has proven difficult to derive an analytical expression for tight bounds on the essential support of (7). However, the bounds depend only on $\angle(M)$, as γ is fixed by the geometric parameter R/D. The essential support of (7) can be numerically evaluated with $|M|=1$ and tabulated over a range of $\angle M$. This table can be used to determine a tight estimate of the support in k for any ω_u,ω_v through the definition of M in (12).

The support in k is the union over all points in the object. However, the support is sufficiently determined by a point on the 'rims' of the cylinder, for example $(R,0,Z)$. This follows from (12), where $|M|$, and therefore the essential support, increases with R and z. Additionally, $|M|$ increases with ω_u, ω_v, so finding the maximum width in k only requires checking frequencies on the circle $\omega_u^2+\omega_v^2 = (\Omega(D+R)/D))^2$.

Defining $\tilde{\Omega} = \Omega(D+R)/D$, the maximum value of $|M|$ is determined from (12) as

$$\begin{aligned}\max|M|^2 &= \max_{\omega_u^2+\omega_v^2=\tilde{\Omega}^2}\left(\frac{1}{2}R(\gamma-\gamma^{-1})\omega_u\right)^2 + (z\omega_v)^2 \\ &= \max_{\omega_u^2+\omega_v^2=\tilde{\Omega}^2}(W_u\omega_u)^2 + (W_v\omega_v)^2 \end{aligned}\tag{14}$$

which is maximized by $\omega_v = 0$ when $W_u^2 > W_v^2$, and $\omega_u = 0$ when $W_v^2 > W_u^2$. In the case of $\omega_v = 0$, the integral in (7) reverts to the 2-D fan beam case, which is known to have $k_{max} = R\Omega$. There are effects of $\angle M$ on the support in k, but as long as $|W_u|$ is substantially larger than $|W_v|$, the bound of $R\Omega$ will be valid. For most imaging scenarios this will be the case, as the required cone angle (in the v dimension) for $|W_v| > |W_u|$ is impractical. For example, a ratio $R/D = 0.5$, representing a fan angle of 60°, would require $Z > 1.7R$ which is a cone angle of 120°. In the case of very large cone and fan angles, then the maximum bounds in k will require checking among the frequencies $\omega_u^2 + \omega_v^2 = \tilde{\Omega}^2$.

IV. Sampling Requirements of $g(u, v, \beta)$

The essential support of G as determined in the previous section provide sampling requirements for g. Assuming a rectangular (separable) sampling lattice in (u, v, β), the conditions to avoid aliasing of the essential support of the FT of the projection data yield the sampling requirements

$$\begin{aligned} T_u, T_v &= \frac{D}{2\Omega(D+R)} \\ T_\beta &= \frac{\pi}{k_{max}} \sim \frac{\pi}{R\Omega} \end{aligned} \tag{15}$$

Since the derivation dealt with bounds in ω_u, ω_v separately from bounds in k, the extent of the region of essential support will be overestimated (as demonstrated next). A derivation that operates jointly over ω_u, ω_v and k is required to provide the tightest estimate. Additionally, other sampling schemes, such as a non-rectangular lattice, may be able to pack the essential support more tightly, resulting in lower sampling requirements.

V. Simulation

The bounds on the essential support of G are verified on a test object. We define a small, spherically symmetric and essentially bandlimited basis function, and create the full object as a superposition of randomly scaled and shifted versions of this basis function. The full object is bounded by a cylinder with $R = 256$mm, $Z = 256$mm, and is spectrally limited to a sphere with $\Omega = 2.5$ cycles/cm. Analytical cone beam projections were simulated using a source distance of D = 640 mm.

Figure 3A shows the essential support of the projection data, and Figure 3B shows the estimated essential support using the bounds in Section III. Figure 4A,B displays some cuts through the essential support volume on a log-compressed scale (darker colors represents larger values) with the estimated bounds indicated by the bold line. The horizontal dashed lines denote the source of the 1-D slices plotted in Figure 4C,D. These 1-D plots through the spectral volume are shown on a logarithmic scale, demonstrating the exponential decay outside of the essential support. The estimated bounds in this case are shown by vertical dashed lines. These figures demonstrate that the derived bounds provide a reasonable estimate of the essential support of G.

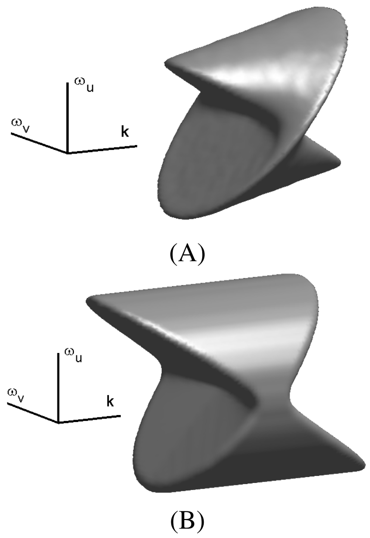

Fig. 3. (A) Measured and (B) estimated essential support

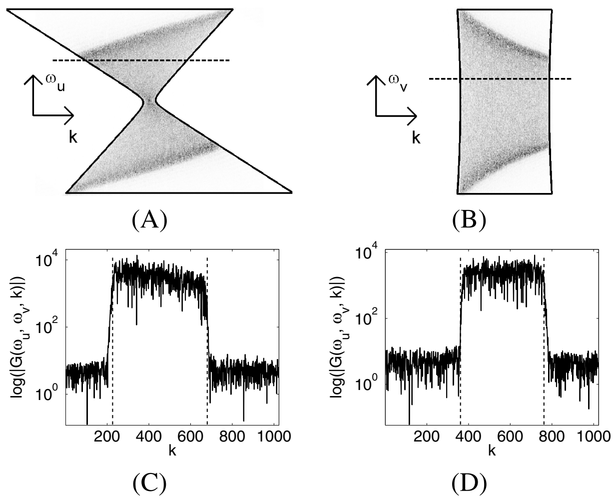

Fig. 4. Slices (A) $\omega_v = 0.46\Omega$, (B) $\omega_u = 0.46\Omega$, (C) $\omega_u = 0.52\Omega$, $\omega_v = 0.46\Omega$ and (D) $\omega_u = 0.46\Omega$, $\omega_v = 0.29\Omega$

VI. Conclusion

In this paper we examined the essential support of cone beam projection data and determined sampling requirements for a CCB scan. This analysis was performed for a flat panel detector and cylindrical object. Future efforts are aimed at a tighter estimate of the essential support for improved sampling conditions. Although conservative, these bounds provide the sampling requirements necessary for alias-free acquisition of the CCB projection data.

References

[1] F. Natterer, Sampling in Fan Beam Tomography. *SIAM Journal Applied Mathematics*, pp. 358-380, 1993.
[2] L. Desbat et al., Sampling Conditions of 3D Parallel and Fan-Beam X-Ray CT with Application to Helical Tomography. *Physics in Medicine and Biology*, v. 49, p. 2377-90, 2004.
[3] J. Brokish, Fast Backprojection Algorithms for Divergent Beam Tomography, Ph.D. Dissertation, University of Illinois at Urbana-Champaign, 2004.